Abstract: The paper investigates regimes of self–excitation in oscillatory systems with attached nonlinear energy sink (NES). For the simple example of primary Van der Pol oscillator, the initial equations are reduced by averaging to 3D system. Small relative mass of the NES justifies analysis of this averaged system as singularly perturbed with two “slow” and one “super–slow” variable. Such approach, in turn, provides complete analytic description of possible response regimes. In addition to almost unperturbed limit cycle oscillations (LCOs), the system can exhibit complete elimination of the self–excitation, small–amplitude LCOs as well as excitation of quasiperiodic strongly modulated response (SMR). In the space of parameters, the latter can be approached by three distinct bifurcation mechanisms: canard explosion, Shil’nikov bifurcation and heteroclinic bifurcation. Some of the above oscillatory regimes can co–exist for the same values of the system parameters. In this case, it is possible to establish the basins of attraction for the co-existing regimes. Direct numeric simulations demonstrate good coincidence with the analytic predictions. More complicated primary systems with internal resonance can give rise to more complicated self–excited responses, including chaotic modulation of the primary frequency.

Keywords: Van der Pol oscillator, nonlinear energy sink, strongly modulated response, global bifurcation, relaxation oscillations, chaotic modulation.

1. INTRODUCTION

Van der Pol (VDP) oscillator is a classical model of system with self–excitation. It has been initially formulated for electric contours [1] and widely used in various phenomenological models involving limit cycle oscillations (LCOs) [2-5]. Nonlinear energy sink (NES) is a relatively recent idea based on attaching a strongly nonlinear (non-linearizable) oscillator to primary mechanical structure. Such structural element has salient dynamical features and can be used in various contexts of vibration protection and mitigation [6-11].

Suppression of the LCO in the VDP oscillator with the NES attached was considered in [12] (see also [11], v. II). In this paper the authors have performed the averaging and analyzed the equations of the averaged flow. They have investigated the fixed points of the slow–flow equations (the latter correspond to steady–state oscillatory regimes in the initial system), studied their stability and bifurcations by means of numeric continuation software.

However, it is well known that such local analysis is, generally speaking, insufficient for complete understanding of the dynamics of excited systems with the NES. In particular, in externally forced linear oscillator with the NES there exists a family of so called strongly modulated responses (SMRs), related to the relaxation oscillations of the averaged flow [13-16]. These attractors can appear or disappear via global bifurcations of the averaged flow, not related to the fixed points and thus untreatable by the local analysis.

The goal of present paper is to apply the methods of global asymptotic analysis of the averaged flow to the VDP oscillator with the NES and to study on this basis the structure of its oscillatory regimes and their bifurcations. The structure of the paper is as follows. In the second Section the model is defined and averaged equations of the slow flow are derived and analyzed. Then we analyze particular examples of the response regimes and their bifurcations, based on the method developed. The last Section contains some concluding remarks and discussion.

2. DESCRIPTION OF THE MODEL

Let us consider common Van der Pol oscillator with attached purely cubic NES II (ungrounded) with linear damping [11]. The initial system of equations describing this system can be written as follows:

\[
\begin{align*}
\frac{d^2}{dt^2}x_1 + c\frac{dx_1}{dt}(x_1^2 - A^2) + q x_1 + \\
+\gamma\left(\frac{d}{dt}x_1 - \frac{dx_2}{dt}\right) + K(x_1 - x_2)^3 &= 0 \\
\frac{d^2}{dt^2}x_2 + \gamma\left(\frac{d}{dt}x_2 - \frac{dx_1}{dt}\right) + K(x_2 - x_1)^3 &= 0
\end{align*}
\]

In this system, \(m_i, i=1,2\) are the masses of the primary oscillator and the NES respectively, \(q\) is the linear stiffness of the VDP oscillator, coefficients \(c\) and \(A\) characterize the combination of positive and negative
damping in the VDP model, γ and K are the damping coefficient and the nonlinear stiffness of the NES respectively. After rescaling, system (1) is reduced to a non-dimensional form with four independent parameters:

\[
\begin{align*}
\ddot{u}_1 + \varepsilon \alpha u_1 (u_1^2 - 1) + u_1 + \varepsilon \lambda (u_1 - u_2) + \frac{4}{3} \varepsilon k (u_1 - u_2)^3 &= 0 \\
\ddot{u}_2 + \lambda (u_2 - u_1) + \frac{4}{3} k (u_2 - u_1)^3 &= 0
\end{align*}
\]

where

\[
\tau = \omega t, \ \omega = \sqrt{m_1}, u_i = x_i / A, \ i = 1, 2,
\]

\[
\varepsilon = m_2 / m_1, \ \alpha = c A^2 / m_1 \omega^2, \ \lambda = \gamma / m_1 \omega^2, k = 3 K A^2 / 4 m_1 \omega^2
\]

the dot denotes the differentiation with respect to τ.

Let us introduce new coordinate, representing a relative displacement in the NES:

\[
w = u_1 - u_2
\]

With account of (3), system (2) is rewritten as follows:

\[
\begin{align*}
\ddot{w} + (1 + \varepsilon) \lambda \dot{w} + \frac{4}{3} \varepsilon k w^3 + \varepsilon \alpha u_1 (u_1^2 - 1) + u_1 &= 0 \\
\dot{w} + (1 + \varepsilon) \lambda \dot{w} + \frac{4}{3} \varepsilon k w^3 + \varepsilon \alpha u_1 (u_1^2 - 1) + u_1 &= 0
\end{align*}
\]

System (4) is the basis for further analysis. We are interested in the motion of the system in the vicinity of 1:1 resonance manifold, where all variables oscillate with frequency close to the natural frequency of the VDP oscillator. Applicability and technicalities of the averaging procedure for this kind of essentially nonlinear systems are discussed elsewhere [17,18]. Change of variables [17] procedure for this kind of essentially nonlinear systems are of parameters the system is still in the state of 1:1 resonance

\[
\begin{align*}
\phi &= \exp(it), \ \xi = \exp(it) \\
\dot{\phi} &= -\varepsilon \alpha \phi \left( \phi \xi^2 / 4 - 1 \right) + \frac{\varepsilon \alpha \xi \dot{\xi}}{2} + i \varepsilon k \dot{\xi} / 2 |\dot{\phi}|^2 - \varepsilon \alpha \phi \left( \phi \xi^2 / 4 - 1 \right) \\
\dot{\xi} &= \frac{i}{2} \left( \phi - \dot{\xi} \right) + \frac{1}{2} (1 + \varepsilon) \lambda \dot{\xi} + \frac{i}{2} (1 + \varepsilon) \lambda \dot{\xi} / 2 |\dot{\phi}|^2 - \varepsilon \alpha \phi \left( \phi \xi^2 / 4 - 1 \right) 
\end{align*}
\]

The averaging presented above has been performed formally beyond the range of applicability of the averaging theory [19]. Still, this procedure was shown to yield rather reliable results in similar systems with the nonlinearizable NES. Anyway, the analytic results should be checked by numeric simulation in order to verify whether for given set of parameters the system is still in the state of 1:1 resonance and the ad hoc averaging performed above can be relied on.

Further reduction of System (6) is performed by splitting the complex variables φ and ξ into modulus and argument parts:

\[
\begin{align*}
\phi &= R \exp(i \delta), \ \xi = P \exp(i \delta), \ \delta = \delta_1 - \delta_2 \\
\dot{R} &= -\frac{\varepsilon \alpha}{2} R \left( R^2 / 4 - 1 \right) - \frac{\varepsilon \lambda}{2} P \cos \delta + \varepsilon k P^3 / 2 \\
\dot{P} &= -\frac{R}{2} \sin \delta - (1 + \varepsilon) \lambda \frac{P}{2} - \frac{\varepsilon \alpha}{2} R \left( R^2 / 4 - 1 \right) \cos \delta \\
\dot{\delta} &= \frac{1}{2} - \frac{(1 + \varepsilon) k P^2}{2} - \frac{R}{2 \sin \delta} + \frac{\varepsilon \lambda P}{R} \sin \delta + \frac{k P^3}{P} \cos \delta - \frac{\varepsilon \alpha R \left( R^2 / 4 - 1 \right) \sin \delta}{P}
\end{align*}
\]

This system of slow – flow equation is equivalent to one presented in [12], with different notations and definitions of particular variables. The only apparent way to treat this complicated nonlinear system of equations analytically is to look for its fixed points and to study their stability and bifurcations. System of transcendental equations describing the fixed points of the flow can be obtained from (7) by nullifying the derivatives:

\[
\begin{align*}
\alpha R_0 \left( R_0^2 / 4 - 1 \right) + \lambda P_0 \sin \delta_0 - k P_0^3 \sin \delta_0 &= 0 \\
R_0 \sin \delta_0 + (1 + \varepsilon) \lambda P_0 + \varepsilon \alpha R_0 \left( R_0^2 / 4 - 1 \right) \sin \delta_0 &= 0 \\
1 - (1 + \varepsilon) k P_0^2 - R_0 \sin \delta_0 + + &\frac{\varepsilon \lambda P_0}{R_0} \sin \delta_0 + \frac{k P_0^3}{P_0} \cos \delta_0 + \frac{\varepsilon \alpha R_0 \left( R_0^2 / 4 - 1 \right) \sin \delta_0}{P_0} = 0
\end{align*}
\]

where the subscript 0 denotes the values of all variables in the fixed points of the system.

System (8) can be reduced to single algebraic equation. In order to see that, let us first write down an equation of energy balance for initial system (2):

\[
\frac{dE}{dt} = \frac{d}{dt} \frac{1}{2} u_1^2 + u_2^2 + \frac{1}{2} \varepsilon k (u_1 - u_2)^2 + \frac{4}{3} \varepsilon k (u_1 - u_2)^3
\]

\[
= -\varepsilon (\alpha \xi u_1 (u_1^2 - 1) + \lambda (u_1 - u_2)^3)
\]

where E is total instantaneous energy of the system. Averaging equation (9) according to (5-7) and substituting zero values for the time derivatives of the slow – flow amplitudes, one obtains:

\[
\lambda P_0 = \alpha R_0 \left( R_0^2 / 4 - 1 \right)
\]

With the help of (10), two first equations of system (8) are reduced to the following form:

\[
-k P_0^2 R_0 \sin \delta_0 + \lambda R_0 \cos \delta_0 = \lambda P_0 R_0
\]

\[
R_0 \sin \delta_0 - \varepsilon \lambda P_0^2 \cos \delta_0 = (1 + \varepsilon) \lambda P_0 R_0
\]

Solving (11) for sinδ₀ and cosδ₀ and using trigonometric identity, we obtain:

\[
\lambda P_0 \left( k (1 + \varepsilon) P_0 - 1 \right) = \\
= \lambda R_0 \left( k P_0 - R_0 \right)^2
\]

Combining (10) with (12), one can obtain the algebraic equation of 6th power with respect to $Q = R_0^2$. We do not bring it here due to its awkwardness. Of course, only real solutions with $Q \geq 0$ should be taken into account. Following steps require numeric solution of this equation and
investigation of stability with the help of linear analysis based on system (7). Such procedure is equivalent to one carried out in [12], although there no reduction to single algebraic equation was performed. Still, at the end one should anyway rely on numeric simulations, so the gain achieved by such reduction is comparatively minor.

The problem is that the system under consideration possesses dynamic regimes which are not related to fixed points of the slow flow (7). Numerous examples of such behavior will be presented below. Treatment of these regimes requires global analysis of system (7); the latter seems to be impossible without further simplifying assumptions.

3. ASYMPTOTIC ANALYSIS IN THE CASE OF LIGHT – WEIGHT NES

It is natural to suggest that the NES has small weight compared to the mass of the principal oscillator. Besides, the averaging procedure performed above has a chance to be valid if the frequency of vibrations is close to the natural frequency of the primary linear oscillator. In other terms, one can adopt the following assumptions concerning the order of magnitude of coefficients in system (7):

$$\varepsilon, \alpha, k, \lambda \sim O(1)$$

Assumptions (13) allow further asymptotic reduction of system (7). For this sake, one can mention that the time derivative in the first equation is of order $O(\varepsilon)$, whereas in two other equations it is of order $O(1)$. Thus, system (7) can be considered as problem of singular perturbation with two "slow" and one "super-slow" variable. The term "slow" is related to the evolution of the averaged flow (7), whereas the term "fast" is reserved for the oscillations which were averaged out. So, the analysis of the initial problem requires consideration of three time scales.

In order to perform the analysis, we introduce the time scales as:

$$\tau_0 = \tau, \tau_1 = \varepsilon \tau$$

At "slow" time scale, system (7) is reduced to the following form:

$$\frac{dR}{d\tau_0} = 0$$

$$\frac{dP}{d\tau_0} = -\frac{R}{2} \sin \delta - \frac{\lambda}{2} P$$

$$\frac{d\delta}{d\tau_0} = \frac{1}{2} - \frac{k}{2} P^2 - \frac{R}{2P} \cos \delta$$

From (15) one easily obtains:

$$\Delta(\tau_0) = -\frac{\lambda N(\tau_0)}{R(\tau_0)}, \quad \cos \Delta(\tau_0) = \frac{N(\tau_0)(1-kN^2(\tau_0))}{R(\tau_0)}$$

In the following we will omit for brevity the explicit dependence on $\tau_1$. Besides, we’ll use the notations

$$\Delta = \frac{N^2(\tau_0)}{R(\tau_0)} + (1-k)^2 \Delta^2$$

In these notations, the last equation of (17) is rewritten as

$$\Delta = \frac{1}{2} \Delta^2 (1-k\Delta)^2$$

This equation determines the slow invariant manifold (SIM) of the problem [16]. Depending on the value of $\lambda$, the SIM can consist of one stable branch (for $\lambda > 1/\sqrt{3}$) or of one unstable and two stable branches (for $\lambda \leq 1/\sqrt{3}$). The latter situation is illustrated at Fig. 1.

Fig. 1. The structure of the slow invariant manifold for $\lambda \leq 1/\sqrt{3}$. The unstable branch is denoted by the dashed line. Special points of the SIM are defined in the text.

The shape of the SIM at Fig. 1 suggests for the global scenario of the relaxation oscillations [21-23]. Namely, the super-slow flow can achieve the fold point $Z_1$ or $Z_2$ where lower (upper) stable branch of the SIM disappears. After that, the flow "jumps" at slow time scale to the remaining stable branch, as it is depicted by dashed arrows at Fig. 1. The "landing" points are $Z_u$ and $Z_d$ respectively. For the SIM analyzed in current problem, the expressions for abscissas for these special points are:

$$Z_{d,12} = 2 \pm \sqrt{1 - 3\lambda^2 \frac{3}{3k}}, \quad Z_{u,d} = 2(1 \pm \sqrt{1 - 3\lambda^2 \frac{3}{3k}})$$

The scenario of the relaxation oscillations described above is similar to strongly modulated response (SMR) revealed earlier in externally forced systems with the NES
Here we'll use the same term. As it was mentioned above, this regime is not related directly to the fixed points of the flow and therefore cannot be revealed by means of the local analysis. Still, the above scenario is incomplete. It can be realized only if the super-slow flow will bring the system to the fold points. To reveal whether it is the case, one should analyze the super-slow flow in the next order of approximation. For this sake, it is enough to take only the first equation of system (7). In the limit $\tau_0 \to \infty$ it yields:

$$\frac{\partial R}{\partial \tau_1} = -\frac{\alpha}{2} R \left(\frac{R^2}{4} - 1 - \frac{\lambda}{2} N \cos \Delta + \frac{k}{2} N^3 \sin \Delta \right)$$

(20)

Substituting equations for $\Delta(\tau_1)$ from (16), multiplying by $R$ and taking into account (18), one obtains:

$$\frac{\partial Y}{\partial \tau_1} = \alpha Y \left(1 - \frac{Y}{4} - \lambda Z\right)$$

(21)

Substituting the expression for the SIM (18) into (21), one obtains the following equation for $Z(\tau_1)$:

$$\frac{\partial Z}{\partial \tau_1} = \frac{Z \left[\alpha (\lambda^2 + (1-kZ)^2) (1 - \frac{Z}{4} (\lambda^2 + (1-kZ)^2)) - \lambda\right]}{1 + \lambda^2 - 4kZ + 3k^2Z^2}$$

(22)

Equation (22) can be solved in quadratures, but it is rather awkward task. In order to analyze the dynamics of the system qualitatively, we need only know how the fixed points of (22) are situated at the SIM. For this sake, we have to find zeros of the numerator in (22). It involves solution of fifth-power algebraic equation, and is rather difficult in parametric form (one has to use elliptic theta-functions etc.). Instead, in the next section we'll present these fixed points graphically, as intersection of the SIM (18) and parabola defined by the right-hand side of (21) equal to zero:

$$\alpha Y \left(1 - \frac{Y}{4} - \lambda Z\right) = 0$$

(23)

Not occasionally, the latter condition in fact coincides with equation (10) for fixed points of the complete slow flow (7). These fixed points in the limit $\epsilon \to 0$ should coincide with the fixed points of super-slow flow (22) and condition (10) does not include $\epsilon$. As for the second condition – equation (18) for the SIM – it also can be obtained from (12) in the limit $\epsilon \to 0$.

4. NUMERIC VERIFICATIONS AND PARTICULAR BIFURCATION SCENARIES.

In this Section we are going to describe various options for the self-excitation regimes of the system under consideration for different values of its parameters. The regimes considered will reveal the relationship between the topology of the fixed points at the SIM and global behavior of the initial system.

4.1 Complete elimination of the LCO

Let us consider the following set of parameters:

$$\alpha = 0.15, \lambda = 0.15, k = 0.36, \epsilon = 0.05$$

(24)

At Fig. 2 we present two curves: the SIM (18), denoted by the thick line, and parabola (23) – by the thin line. They intersect only for $Z=0$ and it is the only stable fixed point of the flow. It corresponds to complete elimination of the LCO.

Let us check this prediction numerically. At Fig. 3 numeric simulation of the initial system (2) is presented. Fig. 3a describes the response of the primary VDP oscillator and Fig 3b – that of the NES (relative displacement $w(t) = u_1(t) - u_2(t)$). Initial conditions for the simulation are specified in the figure caption.

As one can see, the LCO in this case is very efficiently suppressed. More prolonged simulation yields for $u_1$ at $t=5000$ the maximum displacement close to 0.0007. Of course, the actual displacement will never be exactly zero, but still this case can qualify for complete elimination of the LCO.
use higher value of the relative mass \( \varepsilon \), thus achieving lower effective value of \( \alpha \). If by some reason it is not feasible, one should rely on alternative (but less efficient) mechanisms of the LCO suppression.

4.2 Partial LCO suppression

Let us consider the next set of parameters, where the LCO is not eliminated completely:

\[
\alpha = 0.6, \lambda = 0.3, k = 0.4, \varepsilon = 0.05
\] (26)

The diagram of the super-slow flow on the SIM is presented at Fig. 4. One can see that the only stable fixed point corresponds to small – amplitude steady – state oscillations at the lower branch of the SIM. In this case the LCO does not disappear completely but is suppressed partially.

Numeric verification of this regime is presented at Figs. 5 a,b. One can see that the regime of partial LCO suppression is indeed observed. Moreover, the amplitudes both of the primary oscillator and the NES are successfully predicted at Fig. 4 (square roots of the ordinate and the abscissa of the stable fixed point respectively).

4.3 Canard explosion and appearance of the SMR.

Further growth of the parameter \( \alpha \) while the rest are kept constant, changes the situation drastically. For the set

\[
\alpha = 1, \lambda = 0.3, k = 0.4, \varepsilon = 0.05
\] (27)

we obtain the diagram presented at Fig. 6.

Both fixed points at this diagram are unstable and the only possible response regime is the stable SMR. This regime is indeed observed in complete system (2) for this set of parameters. The result of numeric simulations is presented at Figs 7 a, b.
It is interesting to investigate how the transition from the steady-state low-amplitude response to the SMR occurs. If one compares Figure 4 and Figure 6, it is clear that the transition occurs when the stable fixed point crosses the fold to the unstable SIM branch. In the lowest-order approximation used here, the fully developed SMR immediately substitutes the steady-state response. However, initial averaged system (7) is smooth and such singular behavior is impossible in it. Namely, the transition described here is regular Hopf bifurcation and the limit cycle born should have small amplitude. This contradiction arises since the lowest-order approximation is insufficient in the vicinity of the fold. Typical behavior in such situations is so-called canard explosion [24]: the amplitude of the limit cycle grows to full-scale relaxation oscillations with exponentially small variation of the governing parameter. The analysis presented here treats only the shape of the modulation envelope; so, it is instructive to verify whether such a mechanism will be observed for the modulation amplitude in complete dynamic flow (2). In order to see a sharp transition, we use lower value of \( \varepsilon = 0.005 \). Results of numeric simulation are presented at Fig. 8 a-d, the values of parameters are presented in figure captions.

![Fig. 8. Canard explosion of the modulation in the complete flow (2). NES displacement versus time is presented. The parameters are \( \varepsilon = 0.005, \lambda = 0.3, k = 0.4 \). Parameter \( \alpha \) is varied: a) \( \alpha = 0.705 \); b) \( \alpha = 0.708 \); c) \( \alpha = 0.716 \), d) \( \alpha = 0.717 \).](image)

One can see that the Hopf bifurcation of the modulation envelope occurs at about \( \alpha = 0.706 \) and the amplitude of the limit cycle in the modulation amplitude is small, as required. This amplitude steadily grows up to \( \alpha = 0.716 \) and then "exploses" to full-scale SMR at \( \alpha = 0.717 \). This picture completely coincides with the canard explosion scenario mentioned above. Analytic estimation of the transition point based on (18) and (23) yields \( \alpha = 0.716 \), in excellent agreement with the numeric results.

### 4.4 Shil’nikov homoclinic bifurcation of the SMR and co-existence of the SMR and the LCO

Further growth of the parameter \( \alpha \) brings about another bifurcation – a birth of saddle-node pair on the upper branch of the SIM (Figure 9). This Figure is produced for the set of parameters

\[
\alpha = 1.3, \lambda = 0.3, k = 0.4, \varepsilon = 0.05
\]

Figure 9 suggests co-existence of two stable response regimes – high-amplitude LCO oscillations (node at the upper branch) and the SMR. These regimes have well-defined basins of attraction. Namely, the trajectories with all initial conditions above the dashed horizontal line passing through the upper saddle will be attracted to the stable node, and below this line – to the SMR. In order to verify this prediction we simulate the response for two sets of initial conditions, denoted as points 1 and 2 at Fig. 9. The results are presented at Figs. 10 a, b and are in complete agreement with the above prediction.

![Fig. 9. Co-existence of stable SMR and stable LCO regime (set of parameters (28)). The super-slow flow allows the SMR (denoted by dashed arrows and arrows at the SIM). Horizontal dashed line divides between basins of attractions of two stable regimes. Point 1 has coordinates (1,1) Point 2 – (2,2).](image)

![Fig. 10. Relative NES displacement for the case of co-existence, set of parameters (28). Initial conditions correspond to points 1 and 2: a) \( u_{1}(0) = 0, u_{2}(0) = 1, u_{3}(0) = 0, u_{4}(0) = 0 \).](image)
b) \( \alpha_{1}(0) = 0, \dot{u}(0) = 1.4, u_{1}(0) = 0, \dot{u}_{1}(0) = 0 \)

If \( \alpha \) grows even further, the SMR disappears. It happens when the upper saddle crosses the "landing point" of the upper SIM branch \( Z_{u} \). One of possible sets of parameters for this critical situation is

\[
\alpha = 2.318, \lambda = 0.3, k = 0.4 \quad (29)
\]

Corresponding diagram is presented at Fig. 11.

Fig. 11. Homoclinic connection leading to the Shilnikov bifurcation

It is easy to see that in this critical situation the cycle of the relaxation oscillations becomes a homoclinic trajectory of the saddle point at the upper SIM branch. This cycle disappears if the saddle moves downwards on the branch. Such a scenario is equivalent to Shil'nikov homoclinic bifurcation of the limit cycles [25]. In this case, the homoclinic connection is formed in the system and then the limit cycle disappears. Normally, it is not easy to prove that the homoclinic connection exists in a system with dimensionality 3 or higher. In our case, however, for \( \varepsilon \to 0 \) this trajectory can be easily demonstrated. Existence of such connection is preserved under small variations of parameters [21]; therefore we can state that this bifurcation scenario should exist also for \( \varepsilon \) small, but finite. When the SMR approaches the homoclinic connection, one should expect significant growth of the modulation period – at the connection it will be "exactly infinite". Such elongation, observed numerically, will confirm the suggested scenario of the bifurcation.

In order to verify that, we perform the simulation for varying \( \alpha \) and constant parameters \( \lambda = 0.3, k = 0.4, \varepsilon = 0.005 \). Results of the simulation are presented at Fig. 12 a–c.

Tiny variations of \( \alpha \) bring about huge variations of the modulation period, finally it becomes infinitely long. Note that at Fig. 12c we obtain the upper stable point instead of the SMR. So, the simulation confirms the scenario of the Shil'nikov bifurcation. It is worth while mentioning that the bifurcation occurs for \( \alpha \approx 2.1 \), whereas the theory predicts \( \alpha \approx 2.32 \). This discrepancy is already rather significant, but still the qualitative picture is predicted correctly.

4.5 Heteroclinic bifurcation of the SMR

Another scenario of global bifurcation of the SMR is realized for the situation depicted at Fig. 13 a,b for the set of parameters presented in the caption.
The saddle – node bifurcation occurs at the point between $Z_2$ and $Z_u$ and the cycle of the relaxation oscillations "converts" to heteroclinic connection between the node and the saddle. Theoretically, this bifurcation should occur at $\alpha = 4.08$. The expected scenario is also related to growth of the modulation period and is confirmed by simulations presented at Figs 14 a-c. The simulations yield the critical value $\alpha \approx 3.93$, close enough to the theoretical prediction.

Fig. 14. Heteroclinic bifurcation of the SMR. Relative displacement of the NES is computed in the time interval 5000 – 15000, $\lambda = 0.3$, $k = 0.09$, $\epsilon = 0.005$; a) $\alpha = 3.90$; b) $\alpha = 3.91$; c) $\alpha = 3.92$

4.6 Co – existence of two LCOs.

The last case considered here is a co-existence of two LCOs. Such situation exists for the set of parameters

$$\alpha = 6, \lambda = 0.2, k = 0.04 \quad (30)$$

From diagram presented at Fig. 15 one can easily see that two stable steady state LCOs should co-exist. The boundary of their basins of attraction will consist of two horizontal lines leading to the fold points and the unstable branch of the SIM.

For numeric verification we pick two points from zones I and II and simulate the response. The results presented at Figs. 16 a,b confirm the above co – existence scenario (note different amplitudes at Fig. 16 a) and b)).

Fig. 15. Co – existence of two stable steady state LCOs for set of parameters (30). Dashed line denotes the boundary between basins of attraction I and II.

Fig. 16. Relative displacement of the NES for set of parameters (30) and $\epsilon = 0.05$. a) Initial conditions picked in zone I, $u_1(0) = 0, \dot{u}_1(0) = 1, u_2(0) = 0, \dot{u}_2(0) = 0$; b) Initial conditions picked in zone II, $u_1(0) = 0, \dot{u}_1(0) = 5, u_2(0) = 0, \dot{u}_2(0) = 0$.

5. CONCLUDING REMARKS AND DISCUSSION

The methodology developed in this paper paves a way for global asymptotic analysis of the system. Periodic responses, global bifurcations of different types, as well as co-existence and basins of attraction of various self – excitation regimes are revealed. All that is possible due to radical simplifications related to "slow – super-slow" decomposition of the averaged flow. Qualitatively, the coincidence between the theory and numeric simulations is complete. Quantitatively, some deviations in the critical values of parameters (up to 10%) were observed.

Relative simplicity of initial system (2) allowed reduction of the local and the global problem to explicit, although complicated, single variable algebraic and differential equations (10 -12) and (22) respectively. For more complicated models of primary self – excited
oscillator and the NES such reduction may be impossible. However, it should be mentioned that the procedure of analysis presented above does not use these expressions at all. Condition (23) could be obtained from energy conservation (as equivalent equation (10)) and equation (18) – from planar dynamic model of the NES in the state of the resonance [26]. Both procedures may be performed explicitly for wide class of models. It means that the asymptotic procedure presented here can have wider field for application than even local analysis of the complete system like (7), which often requires solution of complicated transcendental equations. Besides, one should expect that the bifurcations presented above will be generic for this sort of systems. Needless to say, the exact values of the intersection points of two algebraic curves of low order may be easily found numerically, if required.

Successful application of the decomposition is based on special asymptotic structure of the averaged flow (7) – two "slow" and one "super-slow" variable. In the case of externally forced systems with the NES there are two variables of each type [16]. This complication yields new types of bifurcations (such as folds of stable – unstable limit cycles of the averaged flow), but also makes global homoclinic and heteroclinic bifurcations described above non – generic. Even higher dimensionality requires further asymptotic reduction to make the problem tractable. For instance, if a primary system has few modes but only one of them becomes unstable, then it may be possible to reduce the problem and to consider only the interaction of this mode with the NES [27]. In the end, one again obtains 3D or 4D slow – flow system, ready for further singular perturbation analysis. It is worth while mentioning here that the "fast – slow" decomposition allows efficient study of other types of bifurcations in high – dimensional dynamical systems [28].

As interesting example of such behavior, we present here a system where the "super – slow" subspace will have dimensionality 3. This is the system of two weakly coupled identical linear oscillators; one of them has the self – excitation term and the other is coupled to the NES. This system is described by the following non – dimensionalized system of equations:

\[
\begin{align*}
\dot{u}_1 + \varepsilon \alpha u_1 (u_1^2 - 1) + u_1 + \varepsilon p(u_1 - u_2) &= 0 \\
\dot{u}_2 + u_2 + \varepsilon p(u_2 - u_1) + \lambda \varepsilon (u_2 - v) + \frac{4}{3} k \varepsilon (u_2 - v)^3 &= 0 \\
\dot{v} + \lambda \varepsilon (v - u_2) + \frac{4}{3} k \varepsilon (v - u_2)^3 &= 0
\end{align*}
\] (31)

The slow invariant manifold will have dimensionality 3 and, consequently, its singularity folds will be generically two – dimensional. Thus, the global responses will be governed by 2D mapping of the fold into itself. Such nonlinear mappings are very likely to have chaotic attractors; consequently, we can expect the response regimes corresponding to chaotically modulated SMRs. As Fig. 17 one numeric example of such behavior is presented.

These results further confirm decisive role of the dimensionality of the super – slow manifold in determining possible system responses and their bifurcations.

REFERENCES.

Bifurcations of Self-Excitation Regimes in Oscillatory Systems with Nonlinear Energy Sink
O.V. Gendelman and Tamir Bar.


