Resumo: In this work, we present adaptive beamformer characteristics using statistical inferences, specifically Bayes’ rule. The beamformer optimization is performed at the uncertainty directions using a minimum mean squared error spatial estimator. We present some preliminary results of the beamformer a posteriori inference.

Palavras-chave: stochastic dynamics, adaptive beamformer, Bayes’ rule.

1. INTRODUCTION

Adaptive beamformer is a method applied to signals in order to improve their resolution by reducing noise and interferences [1]. Its applications are present in several areas, such as in signals from radar, seismometers, sonars, medical images, voice processing and wireless communications.

An adaptive beamformer model is presented in Bell et al. [2] where the authors estimate signal arrival in the presence of interferences. Also using an adaptive method, Nilsen and Hafizovic [3] proposes a similar model. Blomberg et al. [4] present minimum variance measures used for signal analysis. These methods are based on statistical inferences about the signal to be detected.

There are several methods that can be applied to statistical inference problems [5]. One example is the Bayesian approximation, which is an applicable tool for beamformer analysis, such as direction-of-arrival (DOA) estimation, as shown by Bell et al. [1]. Similarly, Kannan and Fitzgerald [6] also use the Bayesian approach for DOA studies.

Bayesian methods include classical estimators such as the maximum a posteriori, maximum-likelihood, minimum mean squared error (MMSE), and minimum mean absolute value error. Bayesian inference is based on the minimization of the Bayes risk functions, given by a posterior unknown parameter model, obtained from observations, and a cost-of-error function [7].

Estimation theory is conceived for the determination of the best estimation of an unknown parameter vector from an observed signal. Also, it can be applied to the recovery of a clean signal from another signal corrupted by noise and distortions. An example is the estimation of basic parameters, such as amplitude, frequency and phase, of a sinusoidal signal with noise [7].

Thus, this Bayesian approach is the focus of our work, we present an analysis of the adaptive beamformer characteristics using the a posteriori information, according to the Bayes’ rule. Using the a priori information, it is possible to obtain the a posteriori probability density function (pdf) of the DOA for an uncertainty region of the signal. Similarly to Bell et al. [1], we aim to apply this method to identify the arrival origins of the interference sources.

Initially, in section Background, we present the concepts of array processing and the optimum and statistical beamformer models. Then, in Methodology, we present the signal matrix and the optimum beamformer analysis procedures using a priori DOA inference. In section Results and Discussion, we show a comparison between simulations with the Bayesian beamformer and a MMSE beamformer for some uncertainty regions. We present our final comments in section Conclusion.

2. BACKGROUND

In this section, we comment about the adaptive beamformer. First, we present the signal array modeling, then the optimum beamformer using the a priori information, and the statistical beamformer using Bayesian inference.

2.1. Signal Array Modeling

The basics of array processing consist in determining the planar energy radiation region of a group of sensors [6], this is the concept of the DOA detection by a sensor array. This array improves the quality of signal reconstruction in the presence of noise and interference.
In this context, an array processing application should consider the array geometry, configuration, spatial and temporal characteristics of the signal processing, array resolution, spatial resolution, and wave fields visualization through sensors aperture [8].

Equation (1) is the output modeling of an array with \( N \) sensors as presented by Tsakalides [9]. These sensors have arbitrary positions and direction characteristics and receive signals from \( M \) narrowband sources with central frequencies equal to \( \omega \) and positions given by \( \theta_1, \theta_2, \ldots, \theta_M \).

In this equation, \( \mathbf{x}(t) = \sum_{i=1}^{M} \mathbf{a}(\theta_i) \cdot y_i(t) + \mathbf{n}(t) \) (1)

In this equation, \( \mathbf{x}(t) = [x_1(t), \ldots, x_N(t)]^T \) is the vector of signals received by the array, \( y_i(t) \) is the signal emitted by the \( i \)th source as received by the first sensor, \( \mathbf{a}(\theta_i) = [1, e^{-j\omega \tau_1(\theta_i)}, \ldots, e^{-j\omega \tau_N(\theta_i)}]^T \) is the array steering vector to the \( \theta_i \) direction, \( \tau_m(\theta) \) is the propagation delay between the first and the \( n \)th sensor for a waveform coming from \( \theta_m \), and \( \mathbf{n}(t) = [n_1(t), \ldots, n_N(t)]^T \) is a noise vector.

The output signal of an array processor that is based on a sum-and-delay beamformer is given by Equation (2) according to Johnson and Dudgeon [10], where \( \mathbf{w}_n \) is the element correspondent weight, \( y_n \) is the signal arriving at each sensor, and \( \delta_n \) is the beamformer element delay.

\[
z(t) = \sum_{n=0}^{N-1} \mathbf{w}_n \cdot y_n(t - \delta_n) \quad (2)
\]

From this equation, one might adjust the delay signal processing in order to improve the desired signal \( z(t) \), a technique that is called stacking [11]. The adjusted \( \delta_n \) values result in array output signal with maximum energy, which is associated to the determination of the DOA region. The spatial energy distribution is used in sensor array analysis. This distribution is represented by the beamformer obtained from the Fourier transform of the array sensor weights \( \mathbf{w}_n \) [12].

This section presented a brief review of fundamental array theory. Next section presents the optimum (ideal) beamformer modeling as a base for the statistical adaptive beamformer.

### 2.2. Optimum Beamformer

Given an array with desired signal \( y_o(n) \), interference signal \( \mathbf{i}(n) \), and thermal noise \( \mathbf{n}(n) \) as shown in Equation (3), where signal \( y_o(n) \) has a defined amplitude given by \( \sigma_s \) and random phase uniformly distributed.

\[
x(n) = y_o(n) + \mathbf{i}(n) + \mathbf{n}(n) \quad (3)
\]

The array interference-plus-noise signal can be written as Equation (4), where \( \mathbf{i}(n) \) and \( \mathbf{n}(n) \) are modeled as stochastic processes with zero mean.

\[
x_{i+n}(n) = \mathbf{i}(n) + \mathbf{n}(n) \quad (4)
\]

The interference has spatial correlation with the signal from the interference angles, while the thermal noise is spatially decorrelated and has power equal to \( \sigma_n^2 \). Thus, the three elements of Equation (3) are mutually decorrelated. Because of this, the array correlation matrix is given by Equation (5), where \( \sigma_s^2 \) is the power of the desired signal with direction \( \theta_s \), and the correlation matrices of the interference and noise are \( \mathbf{R}_i \) and \( \mathbf{R}_n \), respectively.

\[
\mathbf{R}_x = \{\mathbf{x}(n)\mathbf{x}^H(n)\} = N\sigma_s^2 \mathbf{w}(\theta_s) \mathbf{w}^H(\theta_s) + \mathbf{R}_i + \mathbf{R}_n \quad (5)
\]

Similarly, the correlation matrix of the interference-plus-noise signal is given by Equation (6), where \( \mathbf{R}_n = \sigma_n^2 \mathbf{I} \) because of the assumption of the thermal noise decorrelation.

\[
\mathbf{R}_{i+n} = \mathbf{R}_i + \sigma_n^2 \mathbf{I} \quad (6)
\]

After all these considerations, the optimum beamformer is defined in Equation (7), according to Manolakis et al. [12].

\[
\mathbf{w}_o = \frac{\mathbf{R}_i^{-1} \mathbf{a}(\theta_s)}{\mathbf{a}^H(\theta_s) \mathbf{R}_i^{-1} \mathbf{a}(\theta_s)} \quad (7)
\]

The optimum beamformer can be understood as a band-pass filter that passes the signals in the \( \theta_s \) direction and that significantly rejects the interference energy of all the other angles. In next section, we present a modification of the optimum adaptive beamformer: the statistical beamformer.

### 2.3. Statistical Adaptive Beamformer

This statistical beamformer is based on the Bayesian approximation as first presented by Bell et al. [1]. It can also be called Bayesian beamformer and it is obtained from the optimum beamformer using the \textit{a priori} information of the DOA uncertainty. This characteristic modifies the \textit{a posteriori} pdf of the optimum beamformer.

For its definition, we first consider the estimation of a random vector \( \Theta \) given an observation vector \( y \). Then, using the Bayes’ rule, we can obtain \( f(\Theta|y) \), the \textit{a posteriori} pdf of \( \Theta \) given \( y \), Equation (8). In this equation, \( f(\Theta) \) is the \textit{a priori} pdf, \( f(y|\Theta) \) is the likelihood, and \( f(y) \) is the normalization factor.

\[
f(\Theta|y) = \frac{f(y|\Theta)f(\Theta)}{f(y)} \quad (8)
\]

The Bayesian beamformer model uses the MMSE estimation (spatial Wiener filter) of the desired signal, \( y_o \). The known DOA for \( y_o \) is a discrete random variable with a \textit{a priori} pdf \( p(\theta) \), where \( \Theta = \{\theta_1, \ldots, \theta_L\} \) and \( L \) is the number of uncertainty directions. As shown by Bell et al. [1], this MMSE estimation is a conditional mean of \( y_o \) given \( X \), Equation (9), where \( \theta \) is the \textit{a priori} information. The matrix \( X \) is a set of \( K \) snapshots of a data vector \( x(t) \) sampled at \( t_1 \ldots t_K \).

\[
\hat{y}_{MMSE}(t_k) = E\{y_o(t_k)|X\} = E\{E\{y_o(t_k)|X, \theta\}\} \quad (9)
\]

Considering that \( \mathbf{w}_{MS}(\theta) \) is the Wiener spatial filter in the \( \theta_i \) direction, the previous equation can be modified to
Equation (10), where \( p(\theta_i|X) \) is the \textit{a posteriori} pdf of \( \theta \) given the observations.

\[
\hat{y}_{MMSE}(t_k) = \sum_{i=1}^{L} p(\theta_i|X)w_{MS}^H(\theta_i)x(t_k)
\]  

(10)

Therefore, according to Bell et al. [1], the optimum estimator is a beamformer given by Equation (11).

\[
w_B = \sum_{i=1}^{L} p(\theta_i|X)w_{MS}(\theta_i)
\]

(11)

The Bayesian beamformer is the sum of the Wiener spatial filters directed to the set of DOA \( \Theta \). In addition, these filters are combined to the \textit{a priori} probabilities of each direction \( \theta_i \). Then, according to the Bayes’ rule, the \textit{a posteriori} pdf is given by Equation (12), where \( p(X|\theta_i) \) is the pdf of the signals given \( \theta_i \).

\[
p(\theta_i|X) = \frac{p(\theta_i)p(X|\theta_i)}{\sum_{i=1}^{L}p(\theta_i)p(X|\theta_j)} \quad i = 1, 2, \ldots, L
\]

(12)

These concepts of array processing and the optimum beamformer models using Bayesian inference are suitable to signals with DOA waveforms uncertainty.

3. METHODOLOGY

We analyze the Bayesian beamformer inference using simulations in Matlab R2008a. For these simulations, we obtain the signal matrix \( X \), according to Kannan and Fitzgerald [6], as shown in Equation (13).

\[
X = A(\theta)Y + N
\]

(13)

In this equation, \( X \) is a \( N \times K \) matrix, \( A(\theta) \) is the \( N \times M \) DOA matrix of the interferences uncertainty regions, \( Y \) is a \( L \times K \) matrix of spatially distributed signals, and \( N \) is a \( N \times K \) matrix of white decorrelated noise.

Using this signal matrix \( X \), we can apply the conventional and optimum beamformers to an array of sensors. This is shown in Figure 1, where interference signals are present in \( \theta_1 = 45^\circ \) and \( \theta_2 = 40^\circ \), the blue and red lines are the conventional and optimum beamformers respectively. The spatial signal matrix has signal-to-noise ratio of \(-15\,\text{dB}\).

As seen in this figure, the optimum beamformer reduces the interference in \( \theta_1 \) more efficiently. The signal-to-interference-plus-noise ratio (SINR) is significantly worse for the conventional beamformer than for the optimum beamformer at every angle. This interference spreads to the secondary lobes.

For our simulations with the optimum and Bayesian beamformers, we used three directions of interference: \( \theta_1 = -45^\circ \), \( \theta_2 = 20^\circ \), and \( \theta_3 = 65^\circ \). The SINR loss of the beamformers using this configuration is shown in Figure 2.

The Bayesian beamformer is obtained using Equations (11) and (12), where we have chosen a uniform \textit{a priori} pdf equal to \( p(\theta) = 0.14 \). We used seven DOA uncertainty angles evenly distributed in \(-10^\circ \leq \theta_o \leq 10^\circ \). This beamformer response is presented in Figure 3.

\[
\text{Figure 1 – Comparison between the conventional and optimum beamformers applied to an array of sensors with interference signals in } \theta_1 = 45^\circ \text{ and } \theta_2 = 40^\circ .
\]

\[
\text{Figure 2 – SINR loss of the beamformers using three directions of interference: } \theta_1 = -45^\circ , \theta_2 = 20^\circ , \text{ and } \theta_3 = 65^\circ .
\]

\[
\text{Figure 3 – Response of the Bayesian beamformer using uniform } \textit{a priori} \text{ pdf } p(\theta) = 0.14 \text{ and seven DOA uncertainty angles evenly distributed in } -10^\circ \leq \theta_o \leq 10^\circ .
\]

Next section, we present a comparison between the MMSE and Bayesian beamformers in order to show some characteristics of the latter.

4. RESULTS AND DISCUSSION

In order to show the validity of the Bayesian inference for adaptive beamformer applications, in Figure 4, we present a
comparison between the MMSE and Bayesian beamformers for the DOA uncertainty.

Figure 4 – Comparison between the response of the MMSE and Bayesian beamformers using the seven DOA uncertainty angles evenly distributed in $-10^\circ \leq \theta_o \leq 10^\circ$.

An interesting aspect is the approximation of the a posteriori information presented in Figure 5. As can be seen, considering the DOA a priori information of 0.14, the Bayesian beamformer results in a higher a posteriori information.

Figure 5 – Approximation of the DOA a posteriori information compared to the DOA a priori information of 0.14 for the Bayesian beamformer.

5. CONCLUSION

The preliminary study presented herein has shown the characteristics of adaptive beamformers when the DOA a posteriori information is used to improve the array performance. The spatial Wiener filters are the basis of the Bayesian beamformer.

For the presented simulations, we used the DOA uncertainty of the desired signals, but the DOA uncertainty of the interferences should also be studied. That is relevant because, considering the a priori inference, some systems have predetermined characteristics for the interference DOA. Therefore, the Bayesian inference is proven to be viable in such cases what is the aim of our future studies.

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