On a mathematical model for *Aedes aegypti* dispersal dynamics

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Abstract: We revisit some results on *Aedes Aegypti* due to ourself. In particular a new model is discussed from the point of view of symmetry methods.

1 Introduction

According to the World Health Organization (WHO), mosquitoes of *Aedes Aegypti* species transmit the Dengue virus for yellow fever and they are the most important arbovirus to man in the world. For this reason these mosquitoes have been widely studied.

It is possible to find *Aedes Aegypti* mosquitoes around the world, not only in the tropical regions, but also beyond them, reaching temperate climates.

To the best of the author’s knowledge, the first partial differential equation model for *Aedes aegypti* was introduced in [4, 5], which was a *semilinear system* of two partial differential equations.

In a recent study, taking into account some papers (see [6, 7]) concerned with the modeling for *Proteus Mirabilis* bacterial colonies, we introduced a quasilinear system for the dispersal dynamics for the *Aedes aegypti*, see [8]. In the mentioned reference we proposed the system

\[
\begin{cases}
u_t = (u^p u_x)_x - 2\nu u^q u_x + \frac{\gamma}{k} v + \left( \frac{\gamma}{k} - \mu_1 \right) u, \\
v_t = ku + (k - \mu_2 - \gamma)v.
\end{cases}
\]

In the previous equations, $u$ and $v$ are, respectively, non dimensional densities of winged population and aquatic population of mosquitoes and $k$, $\gamma$, $\mu_1$, $\mu_2$ are non-dimensional, in general, positive parameters, and $\nu \geq 0$, see [5]. $k$ denotes the ratio between the constants $\bar{k}_1$ and $\bar{k}_2$, which are, respectively, the carrying capacity related to the amount of *findable* nutrients and the carrying capacity effect depending on the occupation of the available breeders. $\gamma$ denotes the specific rate of maturation of the aquatic form into winged female mosquitoes. $\mu_1$, $\mu_2$ are, respectively, the mortality of winged population and the mortality of aquatic population. $\nu$ denotes a constant velocity flux due to wind currents.
2 Lie Symmetries

System (1) was studied from the point of view of Lie point symmetry theory [8] and the following group classification of such a system was carried out

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( p )</th>
<th>( q )</th>
<th>Generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \forall )</td>
<td>( \forall )</td>
<td>( \forall )</td>
<td>( \partial_x, \partial_t, \partial_x \partial_t, u \partial_u + v \partial_v + A(x,t) \partial_u + A(x,t) \partial_v )</td>
</tr>
<tr>
<td>( \forall )</td>
<td>( \forall )</td>
<td>( \forall )</td>
<td>( \partial_x, \partial_t, u \partial_u + v \partial_v + B(x,t) \partial_u + B(x,t) \partial_v )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( -\frac{4}{3} )</td>
<td>( \forall )</td>
<td>( \partial_x, \partial_t, x^2 \partial_x - 3xu \partial_u - 3xv \partial_v, 2x \partial_x - 3u \partial_u - 3v \partial_v )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( \neq -\frac{4}{3} )</td>
<td>( 0 )</td>
<td>( \partial_x, \partial_t, px \partial_x + 2u \partial_u + 2v \partial_v )</td>
</tr>
<tr>
<td>( \neq 0 )</td>
<td>( 2q )</td>
<td>( \neq 0 )</td>
<td>( \partial_x, \partial_t, qx \partial_x + u \partial_u + v \partial_v )</td>
</tr>
</tbody>
</table>

Tabela 1: Group classification.

The functions \((A, \tilde{A})\) and \((B, \tilde{B})\) appearing in the second and third rows of the Table 1 are, respectively, solutions of the following linear system

\[
\begin{align*}
A_t &= A_{xx} + \frac{\gamma}{k} \tilde{A} + \left( \frac{\gamma}{k} - \mu_1 \right) A, \\
\tilde{A}_t &= kA + (k - \mu_2 - \gamma) \tilde{A}, \\
B_t &= B_{xx} + 2\nu B_x + \frac{\gamma}{k} \tilde{B} + \left( \frac{\gamma}{k} - \mu_1 \right) B, \\
\tilde{B}_t &= kB + (k - \mu_2 - \gamma) \tilde{B}.
\end{align*}
\]

3 Exact solutions

Here, by using some symmetries recalled in the previous section, we derive some exact invariant solutions for two cases of the system under consideration.

3.1 The case \( p = 2q \neq 0 \)

In this case, without loss of generality, the general invariance generator can be written as

\[X = c_3 \partial_t + (qx + c_2) \partial_x + u \partial_u + v \partial_v.\]

From the invariant surface conditions we get

\[
\sigma = e^{\frac{q}{q_3 + c_2}} U, \quad u = e^{-\frac{q}{q_3}} U(\sigma), \quad v = e^{-\frac{q}{q_3}} V(\sigma).
\]

Then after having substituted (2) in the original system (1), by setting \( p = 2q \), we get the following reduced system:

\[
\frac{1}{c_3}(U + q \sigma U') = 2q^2 \sigma^4 U'^{-1}(U')^2 + 2q^2 \sigma^4 U'^{2q} U'' + q^2 \sigma^4 U'^{2q} U'' + 2\nu q^2 \sigma^2 U'' + \frac{\gamma}{k} V + \left( \frac{\gamma}{k} - \mu_1 \right) U,
\]

\[
\frac{1}{c_3}(V + q \sigma V') = kU + (k - \mu_2 - \gamma) V.
\]

In [8] we derived for the system (3) the following solutions

\[U = V_{10} \sigma^{\frac{1}{q}}, \quad V = V_{20} \sigma^{\frac{1}{q}}.\]

Here specializing \( q = 2 \) and going back to the original variables we show the following stationary solution:
\[ u = V_{10}(2x + c_2)^{\frac{1}{2}}, \quad v = V_{20}(2x + c_2)^{\frac{1}{2}}. \]

where
\[ V_{20} = -\frac{k}{k - \mu_2 - \gamma} V_{10}. \quad (5) \]

and \( V_{10} \) is the real positive root of the following algebraic equation
\[ 3V_{10}^{2q} - 2\nu V_{10}^{q} + \left( -\frac{\gamma}{k} \frac{\gamma}{k - \mu_2 - \gamma} + \frac{\gamma}{k} - \mu_1 \right) = 0. \quad (6) \]

Of course in this case it is sufficient to assume that the constitutive parametrs \( k, \gamma, \mu_1, \mu_2, \nu \) satisfy the following inequalities:
\[ k - \mu_2 - \gamma < 0, \quad \nu^2 - 3 \left( \frac{-\gamma}{k} \frac{\gamma}{k - \mu_2 - \gamma} + \frac{\gamma}{k} - \mu_1 \right) \geq 0. \quad (7) \]

### 3.2 Exact solution to the system (1) with \( p = -4/3, \nu = 0 \)

Apart from its mathematical interest this case (a limit case) occurs when it is possible to neglect the wind effects. That is, there is no mosquito migration.

Now consider the linear combination
\[ X = c_1 \partial_x + c_2 \partial_t + c_3 (x^2 \partial_x - 3x u \partial_u - 3x v \partial_v) + c_4 (2x \partial_x - 3u \partial_u - 3v \partial_v) \]

of the Lie point symmetry generators of the system
\[
\begin{cases}
  u_t = (u^{-\frac{4}{3}} u_x)x + \gamma v + \left( \frac{\gamma}{k} - \mu_1 \right) u, \\
  v_t = k u + (k - \mu_2 - \gamma) v.
\end{cases} \quad (8)
\]

Then
\[
\frac{dt}{c_2} = \frac{dx}{c_3 x^2 + 2xc_4 + c_1} = -\frac{du}{3xuc_3 + 3uc_4} = -\frac{dv}{3xvc_3 + 3vjc_4}. \quad (9)
\]

Choosing \( c_1 = a, \ c_1 = a^2, \ c_3 = 1 \) in (9), we obtain
\[
\frac{dt}{c_2} = \frac{dx}{(x + a)^2} = -\frac{du}{3u(x + a)} = -\frac{dv}{3v(x + a)}. \]

From this system we conclude that
\[ \phi = t + \frac{c_2}{x + a}, \quad u = \frac{A(\phi)}{(x + a)^3}, \quad v = \frac{B(\phi)}{(x + a)^3}. \quad (10) \]

Substituting (10) into the original system (8) we arrive at the following ODE system to \( A \) and \( B \):
\[
\begin{align*}
  A' &= \frac{c_2^2}{A^3} \left[ A'' - \frac{4}{3} \left( A' \right)^2 \right] + \gamma B + \left( \frac{\gamma}{k} - \mu_1 \right) A, \\
  B' &= k A + (k - \mu_2 - \gamma) B. \quad (11)
\end{align*}
\]

After having put \( c_2 = 0 \) it is a simple matter to ascertain from (11) that the family of functions
\[
\begin{align*}
  u(x, t) &= \frac{e^{\alpha t}}{(x + a)^3}, \quad v(x, t) = \frac{e^{\alpha t}}{(x + a)^3}
\end{align*}
\]
are solutions to (8) provided that the following compatibility condition holds

\[ 2k^2 + k (\mu_1 - \mu_2 - \gamma) - 2\gamma = 0. \]  

(13)

In fact after substituting (12) in (8) we get that must be:

\[ \alpha = \frac{2\gamma}{k} - \mu_1 = 2k - \mu_2 - \gamma \]

(14)

that implies (13).

These special invariant solutions we derived here, in general, do not satisfy arbitrary initial or boundary conditions prescribed for a given problem. However, they may be useful for a benchmark test for larger numerical schemes devised to solve our system in a realistic case.

We wish to stress that the solutions shown are, until now, the only exact solutions concerned with a mathematical model for \textit{Aedes aegypti} dispersal dynamics. We planned to continue the study [9] about this population by looking for additional exact solutions and further information about constitutive functions appearing in the systems considered.

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Referências


